


1 Robustness for Space-Bounded Statistical Zero 2 Knowledge*

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13 — Abstract —

14 We show that the space-bounded Statistical Zero Knowledge classes SZK_L and NISZK_L are surprisingly
15 robust, in that the power of the verifier and simulator can be strengthened or weakened without
16 affecting the resulting class. Coupled with other recent characterizations of these classes [4], this
17 can be viewed as lending support to the conjecture that these classes may coincide with the
18 non-space-bounded classes SZK and NISZK , respectively.

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1 Introduction

The complexity class SZK (Statistical Zero Knowledge) and its “non-interactive” subclass NISZK have been studied intensively by the research communities in cryptography and computational complexity theory. In [14], a space-bounded version of SZK, denoted SZK_L was introduced, primarily as a tool for understanding the complexity of estimating the entropy of distributions represented by very simple computational models (such as low-degree polynomials, and NC^0 circuits). There, it was shown that SZK_L contains many important problems previously known to lie in SZK, such as Graph Isomorphism, Discrete Log, and Decisional Diffie-Hellman. The corresponding “non-interactive” subclass of SZK_L , denoted NISZK_L , was subsequently introduced in [1], primarily as a tool for clarifying the complexity of computing time-bounded Kolmogorov complexity under very restrictive reducibilities (such as projections). Just as every problem in $\text{SZK} \leq_{\text{tt}}^{\text{AC}^0}$ reduces to problems in NISZK [16], so also every problem in $\text{SZK}_L \leq_{\text{tt}}^{\text{AC}^0}$ reduces to problems in NISZK_L , and thus NISZK_L contains intractable problems if and only if SZK_L does.

Very recently, all of these classes were given surprising new characterizations, in terms of efficient reducibility to the Kolmogorov random strings. Let \tilde{R}_K be the (undecidable) promise problem $(Y_{\tilde{R}_K}, N_{\tilde{R}_K})$ where $Y_{\tilde{R}_K}$ contains all strings y such that $K(y) \geq |y|/2$ and the NO instances $N_{\tilde{R}_K}$ consists of those strings y where $K(y) \leq |y|/2 - e(|y|)$ for some approximation error term $e(n)$, where $e(n) = \omega(\log n)$ and $e(n) = n^{o(1)}$.

► **Theorem 1.** [4] *Let A be a decidable promise problem. Then*

- $A \in \text{NISZK}$ if and only if A is reducible to \tilde{R}_K by randomized polynomial time reductions.
- $A \in \text{NISZK}_L$ if and only if A is reducible to \tilde{R}_K by randomized AC^0 or logspace reductions.
- $A \in \text{SZK}$ if and only if A is reducible to \tilde{R}_K by randomized polynomial time “Boolean formula” reductions.
- $A \in \text{SZK}_L$ if and only if A is reducible to \tilde{R}_K by randomized logspace “Boolean formula” reductions.

In all cases, the randomized reductions are restricted to be “honest”, so that on inputs of length n all queries are of length $\geq n^\epsilon$.

There are very few natural examples of computational problems A where the class of problems reducible to A via polynomial-time reductions differs (or is conjectured to differ) from the class of problems reducible to A via AC^0 reductions. For example the natural complete problems for NISZK under \leq_m^P reductions remain complete under AC^0 reductions. Thus Theorem 1 gives rise to speculation that NISZK and NISZK_L might be equal. (This would also imply that $\text{SZK} = \text{SZK}_L$.)

This motivates a closer examination of SZK_L and NISZK_L , to answer questions that have not been addressed by earlier work on these classes.

Our main results are:

1. **The verifier and simulator may be very weak.** NISZK_L and SZK_L are defined in terms of three algorithms: (1) A logspace-bounded *verifier*, who interacts with (2) a computationally-unbounded *prover*, following the usual rules of an interactive proof, and (3) a logspace-bounded *simulator*, who ensures the zero-knowledge aspects of the protocol. (More formal definitions are to be found in Section 2.) We show that the verifier and simulator can be restricted to lie in AC^0 . Let us explain why this is surprising. The proof presented in [1], showing that EAC^0 is complete for NISZK_L , makes it clear that the verifier and simulator can be restricted to lie in $\text{AC}^0[\oplus]$ (as was observed in [26]).

But the proof in [1] (and a similar argument in [16]) relies heavily on hashing, and it is known that, although there are families of universal hash functions in $AC^0[\oplus]$, no such families lie in AC^0 [21]. We provide an alternative construction, which avoids hashing, and allows the verifier and simulator to be very weak indeed.

2. The verifier and simulator may be somewhat stronger. The proof presented in [1], showing that EA_{NC^0} is complete for $NISZK_L$, also makes it clear that the verifier and simulator can be as powerful as $\oplus L$, without leaving $NISZK_L$. This is because the proof relies on the fact that logspace computation lies in the complexity class $PREN$ of functions that have *perfect randomized encodings* [8], and $\oplus L$ also lies in $PREN$. Applebaum, Ishai, and Kushilevitz defined $PREN$ and the somewhat larger class $SREN$ (for *statistical randomized encodings*), in proving that there are one-way functions in $SREN$ if and only if there are one-way functions in NC^0 . They also showed that other important classes of functions, such as NL and $GapL$, are contained in $SREN$.¹ We initially suspected that $NISZK_L$ could be characterized using verifiers and simulators computable in $GapL$ (or even in the slightly larger class DET , consisting of problems that are $\leq_T^{NC^1}$ reducible to $GapL$), since DET is known to be contained in $NISZK_L$ [1].² However, we were unable to reach that goal.

We were, however, able to show that the simulator and verifier can be as powerful as NL , without making use of the properties of $SREN$. In fact, we go further in that direction. We define the class PM , consisting of those problems that are \leq_T^L -reducible to the Perfect Matching problem. PM contains NL [20], and is not known to lie in (uniform) NC (and it is not known to be contained in $SREN$). We show that statistical zero knowledge protocols defined using simulators and verifiers that are computable in PM yield only problems in $NISZK_L$.

3. The complexity of the simulator is key. As part of our attempt to characterize $NISZK_L$ using simulators and verifiers computable in DET , we considered varying the complexity of the simulator and the verifier separately. Among other things, we show that the verifier can be as complex as DET if the simulator is logspace-computable. In most cases of interest, the $NISZK$ class defined with verifier and simulator lying in some complexity class remains unchanged if the rules are changed so that the verifier is significantly stronger or weaker.

We also establish some additional closure properties of $NISZK_L$ and SZK_L , some of which are required for the characterizations given in [4].

The rest of the paper is organized as follows: Section 3 will show how $NISZK_L$ can be defined equivalently using an AC^0 verifier and simulator. Section 4 will show that increasing the power of the verifier and simulator to lie in PM does not increase the size of $NISZK_L$ (where PM is the class of problems (containing NL) that are logspace Turing reducible to Perfect Matching). Section 5 expands the list of problems known to lie in $NISZK_L$. McKenzie and Cook [22] studied different formulations of the problem of solving linear congruences. These problems are not known to lie in DET , which is the largest well-studied subclass of P known to be contained in $NISZK_L$. However, these problems are randomly logspace-reducible to DET [9]. We show that $NISZK_L$ is closed under randomized logspace reductions, and hence show that these problems also reside in $NISZK_L$. Section 6 shows that the complexity of the simulator is more important than the complexity of the verifier, in non-interactive

¹ This is not stated explicitly for $GapL$, but it follows from [19, Theorem 1]. See also [12, Section 4.2].

² More precisely, as observed in [3], the Rigid Graph (non-) Isomorphism problem is hard for DET [28], and the Rigid Graph Non-Isomorphism problem is in $NISZK_L$ [1, Corollary 23].

zero-knowledge protocols. In particular, the verifier can be as powerful as DET, while still defining only problems in NISZK_L. Finally Section 7 will show that SZK_L is closed under logspace Boolean formula truth-table reductions.

2 Preliminaries

We assume familiarity with the basic complexity classes L, NL, \oplus L and P, and the circuit complexity classes NC⁰ and AC⁰. We assume knowledge of m-reducibility (many-one-reducibility) and Turing-reducibility. #L is the class of functions that count the number of accepting paths of NL machines, and GapL = {f - g : f, g ∈ #L}. The determinant is complete for GapL under $\leq_m^{AC^0}$ ³, and the complexity class DET is the class of languages NC¹-Turing reducible to functions in GapL.

We use the notation $q \sim S$ to denote that element q is chosen uniformly at random from the finite set S .

Many of the problems we consider deal with entropy (also known as Shannon entropy). The *entropy* of a distribution X (denoted $H(X)$) is the expected value of $\log(1/\Pr[X = x])$. Given two distributions X and Y , the *statistical difference* between the two is denoted $\Delta(X, Y)$ and is equal to $\sum_{\alpha} |\Pr[X = \alpha] - \Pr[Y = \alpha]|/2$. Equivalently, for finite domains D , $\Delta(X, Y) = \max_{S \subseteq D} |\Pr_X[S] - \Pr_Y[S]|$. This quantity is also known as the *total variation distance* between X and Y . The *support* of X , denoted $\text{supp}(X)$, is $\{x : \Pr[X = x] > 0\}$.

► **Definition 2.** *Promise Problem:* a promise problem Π is a pair of disjoint sets (Π_Y, Π_N) (the “YES” and “NO” instances, respectively). A solution for Π is any set S such that $\Pi_Y \subseteq S$, and $S \cap \Pi_N = \emptyset$.

► **Definition 3.** A branching program is a directed acyclic graph with a single source and two sinks labeled 1 and 0, respectively. Each non-sink node in the graph is labeled with a variable in $\{x_1, \dots, x_n\}$ and has two edges leading out of it: one labeled 1 and one labeled 0. A branching program computes a Boolean function f on input $x = x_1 \dots x_n$ by first placing a pebble on the source node. At any time when the pebble is on a node v labeled x_i , the pebble is moved to the (unique) vertex u that is reached by the edge labeled 1 if $x_i = 1$ (or by the edge labeled 0 if $x_i = 0$). If the pebble eventually reaches the sink labeled b , then $f(x) = b$. Branching programs can also be used to compute functions $f : \{0, 1\}^m \rightarrow \{0, 1\}^n$, by concatenating n branching programs p_1, \dots, p_n , where p_i computes the function $f_i(x) =$ the i -th bit of $f(x)$. For more information on the definitions, backgrounds, and nuances of these complexity classes, circuits, and branching programs, see the text by Vollmer [29].

► **Definition 4.** Non-interactive zero-knowledge proof (NISZK) [Adapted from [1, 16]]: A non-interactive statistical zero-knowledge proof system for a promise problem Π is defined by a pair of deterministic polynomial time machines⁴ (V, S) (the verifier and simulator, respectively) and a probabilistic routine P (the prover) that is computationally unbounded, together with a polynomial $r(n)$ (which will give the size of the random reference string σ), such that:

1. (Completeness): For all $x \in \Pi_Y$, the probability (over random σ , and over the random choices of P) that $V(x, \sigma, P(x, \sigma))$ accepts is at least $1 - 2^{-O(|x|)}$.

³ See, for instance [6, Theorem 1] for a discussion of the history of this result.

⁴ In prior work on NISZK [16, 1], the verifier and simulator were said to be probabilistic machines. We prefer to be explicit about the random input sequences provided to each machine, and thus the machines can be viewed as deterministic machines taking a sequence of random bits as input.

- 155 2. (Soundness): For all $x \in \Pi_N$, and for every possible prover P' , the probability that
 156 $V(x, \sigma, P'(x, \sigma))$ accepts is at most $2^{-O(|x|)}$. (Note P' here can be malicious, meaning it
 157 can try to fool the verifier)
- 158 3. (Zero Knowledge): For all $x \in \Pi_Y$, the statistical distance between the following two
 159 distributions is bounded by $2^{-|x|}$:
- 160 a. Choose $\sigma \leftarrow \{0, 1\}^{r(|x|)}$ uniformly random, $p \leftarrow P(x, \sigma)$, and output (p, σ) .
 161 b. $S(x, r)$ (where the coins r for S are chosen uniformly at random).

162 It is known that changing the definition, to have the error probability in the soundness and
 163 completeness conditions and in the simulator's deviation be $\frac{1}{n^{\omega(1)}}$ results in an equivalent
 164 definition [1, 16]. (See the comments after [1, Claim 39].) We will occasionally make use of
 165 this equivalent formulation, when it is convenient.

166 NISZK is the class of promise problems for which there is a non-interactive statistical
 167 zero knowledge proof system.

168 NISZK_C denotes the class of problems in NISZK where the verifier V and simulator S lie
 169 in complexity class C .

170 ► **Definition 5.** [1, 16] (EA and EA_{NC⁰}). Consider Boolean circuits $C_X : \{0, 1\}^m \rightarrow \{0, 1\}^n$
 171 representing distribution X . (That is, $\Pr[X = x] = \Pr[C(y) = x]$ where y is chosen uniformly
 172 at random.) The promise problem EA is given by:

$$173 \quad \text{EA}_Y := \{(C_X, k) : H(X) > k + 1\}$$

174

$$175 \quad \text{EA}_N := \{(C_X, k) : H(X) < k - 1\}$$

176 EA_{NC⁰} is the variant of EA where the distribution C_X is an NC⁰ circuit with each output bit
 177 depending on at most 4 input bits.

178 ► **Definition 6** (SDU and SDU_{NC⁰}). Consider Boolean circuits $C_X : \{0, 1\}^m \rightarrow \{0, 1\}^n$
 179 representing distributions X . The promise problem SDU = (SDU_Y, SDU_N) is given by:

$$180 \quad \text{SDU}_Y := \{C_X : \Delta(X, U_n) < 1/n\}$$

181

$$182 \quad \text{SDU}_N := \{C_X : \Delta(X, U_n) > 1 - 1/n\}.$$

183 SDU_{NC⁰} is the analogous problem, where the distributions X are represented by NC⁰ circuits
 184 where no output bit depends on more than four input bits.

185 ► **Theorem 7.** [1, 4]: EA_{NC⁰} and SDU_{NC⁰} are complete for NISZK_L under \leq_m^{proj} . EA_{NC⁰}
 186 remains complete, even if k is fixed to $k = n - 3$.

187 ► **Definition 8.** [14, 27] (SD and SD_{BP}). Consider a pair of Boolean circuits $C_1, C_2 : \{0, 1\}^m \rightarrow \{0, 1\}^n$
 188 representing distributions X_1, X_2 . The promise problem SD is given by:

$$189 \quad \text{SD}_Y := \{(C_1, C_2) : \Delta(X_1, X_2) > 2/3\}$$

190

$$191 \quad \text{SD}_N := \{(C_1, C_2) : \Delta(X_1, X_2) < 1/3\}.$$

192 SD_{BP} is the variant of SD where the distributions X_1, X_2 are represented by branching
 193 programs.

2.1 Perfect Randomized Encodings

We will make use of the machinery of *perfect randomized encodings* [8].

► **Definition 9.** Let $f : \{0, 1\}^n \rightarrow \{0, 1\}^\ell$ be a function. We say that $\hat{f} : \{0, 1\}^n \times \{0, 1\}^m \rightarrow \{0, 1\}^s$ is a *perfect randomized encoding* of f with blowup b if it is:

- **Input independent:** for every $x, x' \in \{0, 1\}^n$ such that $f(x) = f(x')$, the random variables $\hat{f}(x, U_m)$ and $\hat{f}(x', U_m)$ are identically distributed.
- **Output Disjoint:** for every $x, x' \in \{0, 1\}^n$ such that $f(x) \neq f(x')$, $\text{supp}(\hat{f}(x, U_m)) \cap \text{supp}(\hat{f}(x', U_m)) = \emptyset$.
- **Uniform:** for every $x \in \{0, 1\}^n$ the random variable $\hat{f}(x, U_m)$ is uniform over the set $\text{supp}(\hat{f}(x, U_m))$.
- **Balanced:** for every $x, x' \in \{0, 1\}^n$ $|\text{supp}(\hat{f}(x, U_m))| = |\text{supp}(\hat{f}(x', U_m))| = b$

The following property of perfect randomized encodings is established in [14].

► **Lemma 10.** Let $f : \{0, 1\}^n \rightarrow \{0, 1\}^\ell$ be a function and let $\hat{f} : \{0, 1\}^n \times \{0, 1\}^m \rightarrow \{0, 1\}^s$ be a perfect randomized encoding of f with blowup b . Then $H(\hat{f}(U_n, U_m)) = H(f(U_n)) + \log b$.

3 Simulators and Verifiers in AC^0

In this section, we show that NISZK_L can be defined equivalently using verifiers and simulators that are computable in AC^0 . The standard complete problems for NISZK and NISZK_L take a circuit C as input, where the circuit is viewed as representing a probability distribution X ; the goal is to approximate the entropy of X , or to estimate how far X is from the uniform distribution. Earlier work [17, 1, 26] that had presented non-interactive zero-knowledge protocols for these problems had made use of the fact that the verifier could compute hash functions, and thereby convert low-entropy distributions to distributions with small support. But an AC^0 verifier cannot compute hash functions [21].

Our approach is to “delegate” the problem of computing hash functions to a logspace verifier, and then to make use of the uniform encoding of this verifier to obtain the desired distributions via an AC^0 reduction. To this end, we begin by defining a suitably restricted version of SDU_{NC^0} and show that this restricted version remains complete for NISZK_L under AC^0 reductions (and even under projections).⁵

With this new complete problem in hand, we provide a $\text{NISZK}_{\text{AC}^0}$ protocol for the complete problem, to conclude $\text{NISZK}_L = \text{NISZK}_{\text{AC}^0}$.

► **Definition 11.** Consider an NC^0 circuit $C : \{0, 1\}^m \rightarrow \{0, 1\}^n$ and the probability distribution X on $\{0, 1\}^n$ defined as $C(U_m)$ - where U_m denotes m uniformly random bits. For some fixed $\epsilon > 0$ (chosen later in Remark 16), we define:

$$\text{SDU}'_{\text{NC}^0, Y} = \{X : \Delta(C, U_n) < \frac{1}{2^{n^\epsilon}}\}$$

$$\text{SDU}'_{\text{NC}^0, N} = \{X : |\text{supp}(X)| \leq 2^{n-n^\epsilon}\}$$

We will show that $\text{SDU}'_{\text{NC}^0}$ is complete for NISZK_L under uniform \leq_m^{proj} reductions. In order to do so, we first show that $\text{SDU}'_{\text{NC}^0}$ is in NISZK_L by providing a reduction to SDU_{NC^0} .

⁵ This restricted version of SDU_{NC^0} can be seen as a version of the “image density” problem that was defined and studied in [13].

232 \triangleright Claim 12. $\text{SDU}'_{\text{NC}^0} \leq_m^{\text{proj}} \text{SDU}_{\text{NC}^0}$, and thus $\text{SDU}'_{\text{NC}^0} \in \text{NISZK}_L$.

233 **Proof.** On a given probability distribution X defined on $\{0, 1\}^n$ for $\text{SDU}'_{\text{NC}^0}$, we claim that
 234 the identity function $f(X) = X$ is a reduction of $\text{SDU}'_{\text{NC}^0}$ to SDU_{NC^0} . If X is a YES instance
 235 for $\text{SDU}'_{\text{NC}^0}$, then $\Delta(X, U_n) < \frac{1}{2^{n^\epsilon}}$, which clearly is a YES instance of SDU_{NC^0} . If X is a
 236 NO instance for $\text{SDU}'_{\text{NC}^0}$, then $|\text{supp}(X)| \leq 2^{n-n^\epsilon}$. Thus, if we let T be the complement of
 237 $\text{supp}(X)$, we have that, under the uniform distribution, a string α is in T with probability
 238 $\geq 1 - \frac{1}{2^{n^\epsilon}}$, whereas this event has probability zero under X . Thus $\Delta(X, U_n) \geq 1 - \frac{1}{2^{n^\epsilon}}$, easily
 239 making it a NO instance of SDU_{NC^0} . \blacktriangleleft

240 3.1 Hardness for $\text{SDU}'_{\text{NC}^0}$

241 \blacktriangleright **Theorem 13.** $\text{SDU}'_{\text{NC}^0}$ is hard for NISZK_L under \leq_m^{proj} reductions.

242 **Proof.** In order to show that $\text{SDU}'_{\text{NC}^0}$ is hard for NISZK_L , we will show that the reduction
 243 given in [1] proving the hardness of SDU_{NC^0} for NISZK_L actually produces an instance of
 244 $\text{SDU}'_{\text{NC}^0}$.

245 Let Π be an arbitrary promise problem in NISZK_L with proof system (P, V) and simulator
 246 S . Let x be an instance of Π . Let $M_x(r)$ denote a machine that simulates $S(x)$ with
 247 randomness r to obtain a transcript (σ, p) - if $V(x, \sigma, p)$ accepts then $M_x(r)$ outputs σ ; else
 248 it outputs $0^{|\sigma|}$. We will assume without loss of generality that $|\sigma| = n^k$ for some constant k .
 249

250 It was shown in [17, Lemma 3.1] that for the promise problem EA, there is an NISZK
 251 protocol with completeness error, soundness error and simulator deviation all bounded from
 252 above by 2^{-m} for inputs of length m . Furthermore, as noted in the paragraph before Claim
 253 38 in [1], the proof carries over to show that EA_{BP} has an NISZK_L protocol with the same
 254 parameters. Thus, any problem in NISZK_L can be recognized with exponentially small
 255 error parameters by reducing the problem to EA_{BP} and then running the above protocol for
 256 EA_{BP} on that instance. In particular, this holds for EA_{NC^0} . In what follows, let M_x be the
 257 distribution described in the preceding paragraph, assuming that the simulator S and verifier
 258 V yield a protocol with these exponentially small error parameters.

259 \triangleright Claim 14. If $x \in \Pi_{\text{YES}}$ then $\Delta(M_x(r), U_{n^k}) \leq 1/2^{n-1}$. And if $x \in \Pi_{\text{NO}}$ then
 260 $|\text{supp}(M_x(r))| \leq 2^{n^k - n^{\epsilon k}}$ for $\epsilon < \frac{1}{k}$.

261 **Proof.** For $x \in \Pi_{\text{YES}}$, claim 38 of [1] shows that $\Delta(M_x(r), U_{n^k}) \leq 1/2^{n-1}$, establishing the
 262 first part of the claim.

263 For $x \in \Pi_{\text{NO}}$, from the soundness guarantee of the NISZK_L protocol for EA_{NC^0} , we know
 264 that, for at least a $1 - \frac{1}{2^n}$ fraction of the shared reference strings $\sigma \in \{0, 1\}^{n^k}$, there is no
 265 message p that the prover can send that will cause V to accept. Thus there are at most
 266 $2^{n^k - n}$ outputs of $M_x(r)$ other than 0^{n^k} . For $\epsilon < \frac{1}{k}$, we have $|\text{supp}(M_x(r))| \leq 2^{n^k - n^{\epsilon k}}$. \blacktriangleleft

267 The above claim talks about the distribution $M_x(r)$ where M is a logspace machine. We
 268 will instead consider an NC^0 distribution with similar properties that can be constructed
 269 using projections. This distribution (denoted by C_x) is a perfect randomized encoding of
 270 $M_x(r)$. We make use of the following construction:

271 \blacktriangleright **Lemma 15.** [1, Lemma 35]. There is a function computable in AC^0 (in fact, it can be a
 272 projection) that takes as input a branching program Q of size l computing a function f and
 273 produces as output a list p_i of NC^0 circuits, where p_i computes the i -th bit of a function \hat{f}
 274 that is a perfect randomized encoding of f that has blowup $b = 2^{\binom{l}{2} - 1} 2^{((l-1)^2 - 1)}$ (and thus

the length of $\hat{f}(r) = \log b + |f(r)|$. Each p_i depends on at most four input bits from (x, r) (where r is the sequence of random bits in the randomized encoding).

The properties of perfect randomized encodings (see Definition 9) imply that the range of \hat{f} (and thus also the range of C_x) can be partitioned into equal sized pieces corresponding to each value of $f(r)$. Thus, let $\alpha_1, \alpha_2, \dots, \alpha_z$ be the range of $f(r)$, and let $[\alpha] = \{\hat{f}(r, s) : f(r) = \alpha\}$. It follows that $||[\alpha]|| = b$. For a given α , and for a given β of length $\log b$ we denote by $\alpha\beta$ the β -th element of $[\alpha]$. Since the simulator S runs in logspace, each bit of $M_x(r)$ can be simulated with a branching program Q_x . Furthermore, it is straightforward to see that there is an AC^0 -computable function that takes x as input and produces an encoding of Q_x as output, and it can even be seen that this function can be a projection. Let the list of NC^0 circuits produced from Q_x by the construction of Lemma 15 be denoted C_x .

We show that this distribution C_x is an instance of $\text{SDU}'_{\text{NC}^0}$ if $x \in \Pi$. For $x \in \Pi_{YES}$, we have $\Delta(M_x(r), U_{n^k}) \leq 1/2^{n-1}$, and we want to show $\Delta(C_x(r), U_{\log b + n^k}) \leq 1/2^{n-1}$. Thus it will suffice to observe that $\Delta(M_x(r), U_{n^k}) = \Delta(C_x(r), U_{\log b + n^k}) \leq 1/2^{n-1}$.

To see this, note that

$$\begin{aligned} \Delta(C_x(r), U_{\log b + n^k}) &= \sum_{\alpha\beta} \left| \Pr[C_x = \alpha\beta] - \frac{1}{2^{n^k + b}} \right| / 2 = \sum_{\beta} \sum_{\alpha} \left| \Pr[M_x = \alpha] \frac{1}{2^b} - \frac{1}{2^b} \frac{1}{2^{n^k}} \right| / 2 \\ &= \sum_{\alpha} \left| \Pr[M_x = \alpha] - \frac{1}{2^{n^k}} \right| / 2 = \Delta(M_x(r), U_{n^k}). \end{aligned}$$

Thus, for $x \in \Pi_{YES}$, C_x is a YES instance for $\text{SDU}'_{\text{NC}^0}$.

For $x \in \Pi_{NO}$, Claim 14 shows that $|\text{supp}(M_x(r))| \leq 2^{n^k - n}$. Since the NC^0 circuit C_x is a perfect randomized encoding of $M_x(r)$, we have that the size of the support of C_x for $x \in \Pi_{NO}$ is bounded from above by $b \times 2^{n^k - n}$. Note that $\log b$ is polynomial in n ; let $q(n) = \log b$. Let $r(n)$ denote the length of the output of C ; $r(n) = q(n) + n^k$. Thus the size of $\text{supp}(C_x) \leq 2^{n^k - n + q(n)} = 2^{r(n) - n} < 2^{r(n) - r(n)^\epsilon}$ (if $1/\epsilon$ is chosen to be greater than the degree of $r(n)$), and hence C_x is a NO instance for $\text{SDU}'_{\text{NC}^0}$. \blacktriangleleft

► Remark 16. Here is how we pick ϵ in the definition of $\text{SDU}'_{\text{NC}^0}$. SDU_{NC^0} is in NISZK_L via some simulator and verifier, where the error parameters are exponentially small, and the shared reference strings σ have length n^k on inputs of length n . Now we pick $\epsilon > 0$ so that $\epsilon < 1/k$ (as in Claim 14) and also $1/\epsilon$ is greater than the degree of $r(n)$ (as in the last sentence of the proof of Theorem 13).

3.2 $\text{NISZK}_{\text{AC}^0}$ protocol for $\text{SDU}'_{\text{NC}^0}$ on input X represented by circuit C

3.2.1 Non Interactive proof system

1. Let C take inputs of length m and produce outputs of length n , and let σ be the reference string of length n .
2. If there is no r such that $C(r) = \sigma$, then the prover sends \perp . Otherwise, the prover picks an element r uniformly at random from the set $\{r | C(r) = \sigma\}$ and sends it to the verifier.
3. V accepts iff $C(r) = \sigma$. (Since C is an NC^0 circuit, this can be accomplished in AC^0 – this step can not be accomplished in NC^0 since it depends on all of the bits of σ .)

3.2.2 Simulator for $\text{SDU}'_{\text{NC}^0}$ proof system, on input X represented by circuit C

1. Pick a random s of length m and compute $\gamma = C(s)$.
2. Output (s, γ) .

3.3 Proofs of Zero Knowledge, Completeness and Soundness

3.3.1 Completeness

▷ Claim 17. If $X \in \text{SDU}'_{\text{NC}^0, Y}$, then the verifier accepts with probability $\geq 1 - \frac{1}{2^{n^\epsilon}}$.

Proof. If X is a YES instance, then $\Delta(X, U_n) < \frac{1}{2^{n^\epsilon}}$. This implies $|\text{supp}(X)| > 2^n(1 - \frac{1}{2^{n^\epsilon}})$, which immediately implies the stated lower bound on the verifier's probability of acceptance. ◀

3.3.2 Soundness

▷ Claim 18. If $X \in \text{SDU}'_{\text{NC}^0, N}$, then for every prover, the probability that the verifier accepts is at most $\frac{1}{2^{n^\epsilon}}$.

Proof. For every $\sigma \notin \text{supp}(X)$, no prover can make the verifier accept. If $X \in \text{SDU}'_{\text{NC}^0, N}$, the probability that $\sigma \notin \text{supp}(X)$ is greater than $1 - \frac{1}{2^{n^\epsilon}}$. ◀

3.3.3 Statistical Zero-Knowledge

▷ Claim 19. For $X \in \text{SDU}'_{\text{NC}^0, Y}$, $\Delta((p, \sigma), (s, \gamma)) = O(\frac{1}{2^{n^\epsilon}})$.

Proof. Recall that $\sigma \sim \{0, 1\}^n$, $s \sim \{0, 1\}^m$, $p \sim \{r : C(r) = \sigma\}$ and $\gamma = C(s)$. In order to provide an upper bound on $\Delta((p, \sigma), (s, \gamma))$, we consider the element wise probability of each distribution and show that for $X \in \text{SDU}'_{\text{NC}^0, Y}$ the claim holds. For $a \in \{0, 1\}^m$ and $b \in \{0, 1\}^n$ we have :

$$\Delta((p, \sigma), (s, \gamma)) = \sum_{(a, b)} \frac{1}{2} |\Pr[(p, \sigma) = (a, b)] - \Pr[(s, \gamma) = (a, b)]|$$

Let us consider an element $b \in \{0, 1\}^n$. Let $A_b = \{a_1, a_2, \dots, a_{k_b}\}$ be the pre-images of b under C ; that is, for $1 \leq i \leq k_b$ it holds that $C(a_i) = b$. Let $\beta_b = \Pr_{y \sim U_m} [C(y) = b]$. Then $k_b 2^{-m} = \beta_b$ (since exactly k_b elements of $\{0, 1\}^m$ are mapped to b under C). Let $B = \{b \mid \exists y : C(y) = b\}$. Since $\Delta(C(U_m), U_n) \leq \frac{1}{2^{n^\epsilon}}$, it follows that $\frac{|B|}{2^m} \leq \frac{1}{2^{n^\epsilon}}$. We have :

$$\begin{aligned} \Delta((p, \sigma), (s, \gamma)) &= \sum_{(a, b)} \frac{1}{2} (|\Pr[(p, \sigma) = (a, b)] - \Pr[(s, \gamma) = (a, b)]|) \\ &= \frac{1}{2} \sum_{(a, b) : b \in B} |\Pr[(p, \sigma) = (a, b)] - \Pr[(s, \gamma) = (a, b)]| \\ &\quad + \frac{1}{2} \sum_{(a, b) : b \notin B} |\Pr[(p, \sigma) = (a, b)] - \Pr[(s, \gamma) = (a, b)]| \end{aligned}$$

For (a, b) satisfying $b \in B$, we have $\Pr[(s, \gamma) = (a, b)] = \Pr[(p, \sigma) = (a, b)] = 0$. For $b \notin B$ and a satisfying $C(a) \neq b$ we again have $\Pr[(s, \gamma) = (a, b)] = \Pr[(p, \sigma) = (a, b)] = 0$. For (a, b) satisfying $C(a) = b$ we have $\Pr[(s, \gamma) = (a, b)] = 2^{-m}$ since $s \sim U_m$ and picking s fixes b . We also have $\Pr[(p, \sigma) = (a, b)] = \frac{2^{-n}}{k_b}$ since $\sigma \sim U_n$ and then the prover picks p uniformly from

343 A_b . This gives us

$$\begin{aligned}
 344 \quad \Delta((p, \sigma), (s, \gamma)) &= \frac{1}{2} \sum_{(a,b): C(a)=b} \left| 2^{-m} - \frac{2^{-n}}{k_b} \right| \\
 345 \quad &= \frac{1}{2} \sum_{(a,b): C(a)=b} \left| 2^{-m} - \frac{2^{-m-n}}{\beta_b} \right| \\
 346 \quad &= \frac{1}{2} \sum_{(a,b): C(a)=b} \frac{2^{-m}}{\beta_b} |\beta_b - 2^{-n}| \\
 347 \quad &\leq \frac{1}{2} \sum_{(a,b): C(a)=b} |\beta_b - 2^{-n}| = \Delta(C(U_m), U_n) \leq \frac{1}{2^{n^\epsilon}} \\
 348
 \end{aligned}$$

349 where the first inequality holds since $\beta_b \geq 2^{-m}$ whenever $\beta_b \neq 0$. Thus we have :

$$350 \quad \Delta((p, \sigma), (s, \gamma)) = O\left(\frac{1}{2^{n^\epsilon}}\right).$$

351

352 **4 Simulator and Verifier in PM**

353 In this section, we show that NISZK_L can be defined equivalently using verifiers and simulators
 354 that lie in the class PM of problems that logspace-Turing reduce to Perfect Matching. (PM
 355 is not known to lie in (uniform) NC.) That is, we can increase the computational power of the
 356 simulator and the verifier from L to PM without affecting the power of noninteractive
 357 statistical zero knowledge protocols.

358 The Perfect Matching problem is the well-known problem of deciding, given an undirected
 359 graph G with $2n$ vertices, if there is a set of n edges covering all of the vertices. We define a
 360 corresponding complexity class PM as follows:

$$361 \quad \text{PM} := \{A : A \leq_T^L \text{Perfect Matching}\}$$

362 It is known that $\text{NL} \subseteq \text{PM}$ [20].

363 Our argument proceeds by first observing⁶ that $\text{NISZK}_L = \text{NISZK}_{\oplus L}$, and then making
 364 use of the details of the argument that Perfect Matching is in $\oplus L/\text{poly}$ [7].

365 **► Proposition 20.** $\text{NISZK}_{\oplus L} = \text{NISZK}_L$

366 **Proof.** It suffices to show $\text{NISZK}_{\oplus L} \subseteq \text{NISZK}_L$. We do this by showing that the problem
 367 EA_{NC^0} is hard for $\text{NISZK}_{\oplus L}$; this suffices since EA_{NC^0} is complete for NISZK_L . The proof
 368 of [1, Theorem 26] (showing that EA_{NC^0} is complete for NISZK_L involves (a) building a
 369 branching program to simulate a logspace computation called M_x that is constructed from a
 370 logspace-computable simulator and verifier, and (b) constructing an NC^0 -computable perfect
 371 randomized encoding of M_x , using the fact that $L \subset \text{PREN}$, where PREN is the class
 372 defined in [8], consisting of all problems with perfect randomized encodings. But Theorem
 373 4.18 in [8] shows the stronger result that $\oplus L$ lies in PREN , and hence the argument of
 374 [1, Theorem 26] carries over immediately, to reduce any problem in $\text{NISZK}_{\oplus L}$ to EA_{NC^0} (by
 375 modifying step (a), to build a *parity* branching program for M_x that is constructed from a
 376 $\oplus L$ simulator and verifier). ◀

⁶ This equality was previously observed in [26].

377 We also rely on the following lemma:

378 ► **Lemma 21.** *Adapted from [7, Section 3] and [23, Section 4]: Let $W = (w_1, w_2, \dots, w_{n^{k+3}})$*
 379 *be a sequence of n^{k+3} weight functions, where each $w_i : \binom{[n]}{2} \rightarrow [4n^2]$ is a distinct weight*
 380 *assignment to edges in n -vertex graphs. Let (G, w_i) denote the result of weighting the edges*
 381 *of G using weight assignment w_i . Then there is a function f in GapL , such that, if (G, w_i)*
 382 *has a unique perfect matching of weight j , then $f(G, W, i, j) \in \{1, -1\}$, and if G has no*
 383 *perfect matching, then for every (W, i, j) , it holds that $f(G, W, i, j) = 0$. Furthermore, if W*
 384 *is chosen uniformly at random, then with probability $\geq 1 - 2^{-n^k}$, for each n -vertex graph G :*

- 385 ■ *If G has no perfect matching then $\forall i \forall j f(G, W, i, j) = 0$.*
- 386 ■ *If G has a perfect matching then $\exists i$ such that (G, w_i) has a unique minimum-weight*
 387 *matching, and hence $\exists i \exists j f(G, W, i, j) \in \{1, -1\}$.*

388 Thus if we define $g(G, W)$ to be $1 - \prod_{i,j} (1 - f(G, W, i, j)^2)$, we have that $g \in \text{GapL}$ and with
 389 probability $\geq 1 - 2^{-n^k}$ (for randomly-chosen W), $g(G, W) = 1$ if G has a perfect matching,
 390 and $g(G, W) = 0$ otherwise.

391 Note that this lemma is saying that most W constitute a good “advice string”, in the sense
 392 that $g(G, W)$ provides the correct answer to the question “Does G have a perfect matching?”
 393 for every graph G with n vertices.

394 ► **Corollary 22.** *For every language $A \in \text{PM}$ there is a language $B \in \oplus\text{L}$ such that, if $x \in A$,*
 395 *then $\Pr_{W \leftarrow [4n^2]^{n^5}} [(x, W) \in B] \geq 1 - 2^{-n^2}$, and if $x \notin A$, then $\Pr_{W \leftarrow [4n^2]^{n^5}} [(x, W) \in B] \leq$*
 396 *2^{-n^2} .*

397 **Proof.** Let A be in PM , where there is a logspace oracle machine M accepting A with an
 398 oracle P for Perfect Matching. We may assume without loss of generality that all queries
 399 made by M on inputs of length n have the same number of vertices $p(n)$. This is because G
 400 has a perfect matching iff $G \cup \{x_1 - y_1, x_2 - y_2, \dots, x_k - y_k\}$ has a perfect matching. (I.e., we
 401 can “pad” the queries, to make them all the same length.)

402 Let $C = \{(G, W) : g(G, W) \equiv 1 \pmod{2}\}$, where g is the function from Lemma 21. Clearly,
 403 $C \in \oplus\text{L}$. Now, a logspace oracle machine with input (x, W) and oracle C can simulate
 404 the computation of M^P on x ; each time M poses the query “Is $G \in P$ ”, instead we ask if
 405 $(G, W) \in C$. Then with high probability (over the random choice of W) all of the queries
 406 will be answered correctly and hence this routine will accept if and only if $x \in A$, by
 407 Lemma 21. Let B be the language accepted by this logspace oracle machine. We see that
 408 $B \in \text{L}^C \subseteq \text{L}^{\oplus\text{L}} = \oplus\text{L}$, where the last equality is from [18]. ◀

409 ► **Theorem 23.** $\text{NISZK}_\text{L} = \text{NISZK}_{\text{PM}}$

410 **Proof.** We show that $\text{NISZK}_{\text{PM}} \subseteq \text{NISZK}_{\oplus\text{L}}$, and then appeal to Proposition 20.

411 Let Π be an arbitrary problem in NISZK_{PM} , and let (S, P, V) be the PM simulator, prover,
 412 and verifier for Π , respectively. Let S' and V' be the $\oplus\text{L}$ languages that are probabilistic
 413 realizations of S, V , respectively, guaranteed by Corollary 22. We now define a NISZK_L
 414 protocol (S'', P'', V'') for Π .

415 On input x with shared randomness σW , the prover P'' sends the same message $p =$
 416 $P(x, \sigma)$ as the original prover sends. The verifier V'' , returns the value of $V'((x, \sigma, p), W)$,
 417 which with high probability is equal to $V(x, \sigma, p)$. The simulator S'' , given as input x and
 418 random sequence rW , executes $S'((x, r, i), W)$ for each bit position i to obtain a bit that
 419 (with high probability) is equal to the i^{th} bit of $S(x, r)$, which is a string of the form (σ, p) ,
 420 and outputs $(\sigma W, p)$.

421 Now we will analyze the properties of (S'', P'', V'') :

422 ■ Completeness: Suppose $x \in \Pi_Y$, then $\Pr_\sigma[V(x, \sigma, P(x, \sigma)) = 1] \geq 1 - 2^{-O(n)}$. Since
 423 $\forall y \in \{0, 1\}^n : \Pr_W[V(y) = V'(y, W)] \geq 1 - 2^{-n^k}$ we have:

$$424 \Pr_{\sigma W}[V'((x, \sigma, P''(x, \sigma)), W) = 1] \geq [1 - 2^{-O(n)}][1 - 2^{-n^k}] = 1 - 2^{-O(n)}$$

425 ■ Soundness: Suppose $x \in \Pi_N$, then $\Pr_\sigma[\forall p : V(x, \sigma, p) = 0] \geq 1 - 2^{-O(n)}$. Since
 426 $\forall y \in \{0, 1\}^n : \Pr_W[V(y) = V'(y, W)] \geq 1 - 2^{-n^k}$, we have:

$$427 \Pr_{\sigma W}[\forall p : V'((x, \sigma, p), W) = 0] \geq [1 - 2^{-O(n)}][1 - 2^{-n^k}] = 1 - 2^{-O(n)}$$

428 ■ Statistical Zero-Knowledge: Suppose $x \in \Pi_Y$. Let S^* denote the distribution on strings
 429 of the form (σ, p) that $S(x, r)$ produces, where r is uniformly generated, and let P^* denote
 430 the distribution on strings given by $(\sigma, P(x, \sigma))$ where σ is chosen uniformly at random.
 431 Similarly, let S''^* denote the distribution on strings of the form $(\sigma W, p)$ that $S''(x, rW)$
 432 produces, where r and W are chosen uniformly, and let P''^* be the distribution given by
 433 $(\sigma W, P''(x, \sigma W))$. Let $A = \{(\sigma W, p) : \exists i \exists r S(x, r)_i \neq S'((x, r, i), W)\}$.
 434 Since $\Pr_W[\forall i \forall r : S(x, r)_i = S'((x, r, i), W)] \geq 1 - 2^{-O(n)}$ we have:

$$\begin{aligned} 435 \Delta(S''^*, P''^*) &= \frac{1}{2} \sum_{(\sigma W, p)} |\Pr[S''^* = (\sigma W, p)] - \Pr[P''^* = (\sigma W, p)]| \\ 436 &\leq \frac{1}{2}(2^{-O(n)} + \sum_{(\sigma W, p) \in \bar{A}} |\Pr[S''^* = (\sigma W, p)] - \Pr[P''^* = (\sigma W, p)]|) \\ &= \frac{1}{2}(2^{-O(n)} + \sum_{(\sigma W, p) \in \bar{A}} |\Pr[S^* = (\sigma, p)] - \Pr[P^* = (\sigma, p)]| \Pr[W]) \\ 437 &\leq 2^{-O(n)} + \sum_W \Pr[W] \frac{1}{2} \sum_{(\sigma, p)} |\Pr[S^* = (\sigma, p)] - \Pr[P^* = (\sigma, p)]| \\ 438 &= 2^{-O(n)} + \Delta(S^*, P^*) = 2^{-O(n)} \end{aligned}$$

441 Therefore (S'', P'', V'') is a $\text{NISZK}_{\oplus \mathbb{L}}$ protocol deciding Π . ◀

442 5 Additional problems in $\text{NISZK}_{\mathbb{L}}$

443 In this section, we give additional examples of problems in P that lie in $\text{NISZK}_{\mathbb{L}}$. These
 444 problems are not known to lie in (uniform) NC . Our main tool is to show that $\text{NISZK}_{\mathbb{L}}$ is
 445 closed under a class of randomized reductions.

446 The following definition is from [4]:

447 ► **Definition 24.** A promise problem $A = (Y, N)$ is \leq_m^{BPL} -reducible to $B = (Y', N')$ with
 448 threshold θ if there is a logspace-computable function f and there is a polynomial p such that

- 449 ■ $x \in Y$ implies $\Pr_{r \in \{0, 1\}^{p(|x|)}} [f(x, r) \in Y'] \geq \theta$.
- 450 ■ $x \in N$ implies $\Pr_{r \in \{0, 1\}^{p(|x|)}} [f(x, r) \in N'] \geq \theta$.

451 Note, in particular, that the logspace machine computing the reduction has two-way access
 452 to the random bits r ; this is consistent with the model of probabilistic logspace that is used
 453 in defining $\text{NISZK}_{\mathbb{L}}$.

454 ► **Theorem 25.** $\text{NISZK}_{\mathbb{L}}$ is closed under \leq_m^{BPL} reductions with threshold $1 - \frac{1}{n^{\omega(1)}}$.

455 **Proof.** Let $\Pi \leq_m^{\text{BPL}} \text{EA}_{\text{NC}^0}$, via logspace-computable function f . Let (S_1, V_1, P_1) be the NISZK_L
 456 proof system for EA_{NC^0} .

■ **Algorithm 1** Simulator $S(x, r\sigma')$

457 $(\sigma, p) \leftarrow S_1(f(x, \sigma'), r);$
return $((\sigma, \sigma'), p);$

■ **Algorithm 2** Verifier

$V(x, (\sigma, \sigma'), p)$
return $V_1((f(x, \sigma'), \sigma, p))$

■ **Algorithm 3** Prover $P(x, (\sigma, \sigma'))$

458 **return** $P_1((f(x, \sigma'), \sigma));$

459 We now claim that (S, P, V) is a NISZK_L protocol for Π .

460 It is apparent that S and V are computable in logspace. We just need to go through
 461 completeness, soundness, and statistical zero-knowledge of this protocol.

462 ■ Completeness: Suppose x is YES instance of Π . Then with probability $1 - \frac{1}{n^{\omega(1)}}$ (over
 463 randomness of σ'), we have that $f(x, \sigma')$ is a YES instance of EA_{NC^0} . Thus for a randomly
 464 chosen σ :

$$465 \Pr[V_1(f(x, \sigma'), \sigma, P_1(f(x, \sigma'), \sigma)) = 1] \geq 1 - \frac{1}{n^{\omega(1)}}$$

466 ■ Soundness: Suppose x is NO instance of Π . Then with probability $1 - \frac{1}{n^{\omega(1)}}$ (over
 467 randomness of σ'), we have that $f(x, \sigma')$ is a NO instance of EA_{NC^0} . Thus for a randomly
 468 chosen σ :

$$469 \Pr[V_1(f(x, \sigma'), \sigma, P_1(f(x, \sigma'), \sigma)) = 0] \geq 1 - \frac{1}{n^{\omega(1)}}$$

470 ■ Statistical Zero-Knowledge: If x is a YES instance, $f(x, \sigma')$ is a YES instance of EA_{NC^0}
 471 with probability close to 1. For any YES instance y of EA_{NC^0} , the distribution given by
 472 S_1 on input y is exponentially close to the distribution on transcripts (σ, p) induced by
 473 (V_1, P_1) on input y . Thus the distribution on $(\sigma\sigma', p)$ induced by (V, P) has distance at
 474 most $\frac{1}{n^{\omega(1)}}$ from the distribution produced by S on input x . The claim now follows by
 475 the comments regarding error probabilities in Definition 4.

476 ◀

477 McKenzie and Cook [22] defined and studied the problems LCON , LCONX and LCONNUL .
 478 LCON is the problem of determining if a system of linear congruences over the integers mod
 479 q has a solution. LCONX is the problem of finding a solution, if one exists, and LCONNUL
 480 is the problem of computing a spanning set for the null space of the system.

481 These problems are known to lie in uniform NC^3 [22], but are not known to lie in uniform
 482 NC^2 , although Arvind and Vijayaraghavan showed that there is a set B in $\text{L}^{\text{GapL}} \subseteq \text{DET} \subseteq \text{NC}^2$
 483 such that $x \in \text{LCON}$ if and only if $(x, W) \in B$, where W is a randomly-chosen weight function
 484 [9]. (The probability of error is exponentially small.) The mapping $x \mapsto (x, W)$ is clearly a
 485 \leq_m^{BPL} reduction. Since $\text{DET} \subseteq \text{NISZK}_L$ [1], it follows that

486 $\text{LCON} \in \text{NISZK}_L$

487 The arguments in [9] carry over to LCONX and LCONNUL as well.

488 ► **Corollary 26.** $\text{LCON} \in \text{NISZK}_L$. $\text{LCONX} \in \text{NISZK}_L$. $\text{LCONNUL} \in \text{NISZK}_L$.

6 Varying the Power of the Verifier

In this section, we show that the computational complexity of the simulator is more important than the computational complexity of the verifier, in non-interactive protocols. The results in this section were motivated by our attempts to show that $\text{NISZK}_L = \text{NISZK}_{\text{DET}}$. Although we were unable to reach this goal, we were able to show that the verifier could be as powerful as DET , if the simulator was restricted to be no more powerful than NL . The general approach here is to replace a powerful verifier with a weaker verifier, by requiring the prover to provide a proof to convince a weak verifier that the more powerful verifier would accept.

We define $\text{NISZK}_{A,B}$ as the class of problems with a NISZK protocol where the simulator is in A and the verifier is in B (and hence $\text{NISZK}_A = \text{NISZK}_{A,A}$). We will consider the case where $A \subseteq B \subseteq \text{NISZK}_A$ and A, B are both classes of functions that are closed under composition.

Theorem 27. $\text{NISZK}_{A,B} = \text{NISZK}_A$

Proof. Let Π be an arbitrary promise problem in $\text{NISZK}_{A,B}$ with (S_1, V_1, P_1) being the A simulator, B verifier, and prover for Π 's proof system, where the reference string has length $p_1(|x|)$ and the prover's messages have length $q_1(|x|)$. Since $V_1 \in B \subseteq \text{NISZK}_A$, $L(V_1)$ has a proof system (S_2, V_2, P_2) , where the reference string has length $p_2(|x|)$ and the prover's messages have length $q_2(|x|)$.

Lemma 28. We may assume without loss of generality that $p_1(n) > p_2(n) + q_2(n)$.

Proof. If it is not the case that $p_1(n) > p_2(n) + q_2(n)$, then let $r(n) = p_2(n) + q_2(n) - p_1(n)$. Consider a new proof system (S'_1, V'_1, P'_1) that is identical to (S_1, V_1, P_1) , except that the reference string now has length $p_1(n) + r(n)$ (where P'_1 and V'_1 ignore the additional $r(n)$ random bits). The simulator S'_1 uses an additional $r(n)$ random bits and simply appends those bits to the output of S_1 . The language $L(V'_1)$ is still in NISZK_A , with a proof system (S'_2, V'_2, P'_2) where the reference string still has length $p_2(n)$, since membership in $L(V'_1)$ does not depend on the “new” $r(n)$ random bits, and hence S'_2, V'_2 and P'_2 , given input $(x, \sigma r, p)$ behave exactly as S_2, V_2 and P_2 behave when given input (x, σ, p) . ◀

Then Π has the following NISZK_A proof system:

Algorithm 4 Simulator

$S(x, r_1, r_2)$

Data: $x \in \Pi_{Yes} \cup \Pi_{No}$

$(\sigma, p) \leftarrow S_1(x, r_1);$

$(\sigma', p') \leftarrow S_2((x, \sigma, p), r_2);$

return $((\sigma, \sigma'), (p, p'));$

Algorithm 5 Verifier

$V(x, (\sigma, \sigma'), (p, p'))$

return $V_2((x, \sigma, p), \sigma', p')$

Algorithm 6 Prover $P(x, \sigma \sigma')$

Data: $x \in \Pi_{Yes} \cup \Pi_{No}, \sigma \in \{0, 1\}^{p_1(|x|)}, \sigma' \in \{0, 1\}^{p_2(|x|)}$

if $x \in \Pi_{Yes}$ **then**

$p \leftarrow P_1(x, \sigma);$

$p' \leftarrow P_2((x, \sigma, p), \sigma');$

return $(p, p');$

else

return $\perp, \perp;$

end

519 **Correctness:** Suppose $x \in \Pi_{Yes}$, then given random σ , with probability $(1 - \frac{1}{2^{O(|x|)}})$, we
 520 have that $(x, \sigma, P_1(x, \sigma)) \in L(V_1)$, which means with probability $(1 - \frac{1}{2^{O(|x|+p_1(|x|)+|p|)}})$ it
 521 holds that $((x, \sigma, p), \sigma', P_2(x, \sigma, P_1(x, \sigma))) \in L(V_2)$. So the probability that V accepts is
 522 at least:

$$523 \quad (1 - \frac{1}{2^{O(|x|)}})(1 - \frac{1}{2^{O(|x|+p_1(|x|)+q_1(|x|))}}) = 1 - \frac{1}{2^{O(|x|)}}$$

524 **Soundness:** Suppose $x \in \Pi_N$. When given a random σ , we have that with probability less
 525 than $\frac{1}{2^{O(|x|)}}$: $\exists p$ such that $(x, \sigma, p) \in L(V_1)$. For $(x, \sigma, p) \notin L(V_1)$, the probability that
 526 there is a p such that $((x, \sigma, p), \sigma', p') \in L(V_2)$ is at most $\frac{1}{2^{O(|x|+p_1(|x|)+|p|)}}$ (given random
 527 σ'). So the probability that V rejects is at least:

$$528 \quad (1 - \frac{1}{2^{O(|x|)}})(1 - \frac{1}{2^{O(|x|+p(|x|)+|p|)}}) = 1 - \frac{1}{2^{O(|x|)}}$$

529 **Statistical Zero-Knowledge:** Let P_1^* denote the distribution that samples σ and outputs
 530 $(\sigma, P_1(x, \sigma))$. Similarly, let $P_2^*(\sigma, p)$ denote the distribution that samples σ' and outputs
 531 $(\sigma\sigma', P_2((x, \sigma, p), \sigma'))$. P^* will be defined as the distribution $((\sigma\sigma'), P(x, \sigma, \sigma'))$ where σ
 532 and σ' are chosen uniformly at random. In the same way, let S^* refer to the distribution
 533 produced by S on input x , let S_1^* refer to the distribution produced by $S_1(x)$, and let
 534 $S_2^*(\sigma, p)$ be the distribution induced by S_2 on input (x, σ, p) . Now we can partition the
 535 set of possible outcomes $((\sigma, \sigma'), (p, p'))$ of S^* and P^* into 3 blocks:

- 536 1. $((\sigma, \sigma'), (p, p'))$ such that $V_1(x, \sigma, p)$ accepts and $V_2((x, \sigma, p), \sigma', p')$ accepts.
- 537 2. $((\sigma, \sigma'), (p, p'))$ such that $V_1(x, \sigma, p)$ accepts and $V_2((x, \sigma, p), \sigma', p')$ rejects.
- 538 3. $((\sigma, \sigma'), (p, p'))$ such that $V_1(x, \sigma, p)$ rejects.

539 We will call these blocks A_1, A_2 , and A_3 respectively. Then by definition:

$$540 \quad \Delta(S^*, P^*) = \frac{1}{2} \sum_{j \in \{1, 2, 3\}} \sum_{y \in A_j} |\Pr_{S^*}[y] - \Pr_{P^*}[y]|$$

$$541 \quad = \frac{1}{2} \sum_{y \in A_1} |\Pr_{S^*}[y] - \Pr_{P^*}[y]| + \frac{1}{2} \sum_{j \in \{2, 3\}} \sum_{y \in A_j} [\Pr_{S^*}[y] + \Pr_{P^*}[y]]$$

543 We concentrate first on A_1 .

$$544 \quad \sum_{y \in A_1} |\Pr_{S^*}[y] - \Pr_{P^*}[y]|$$

$$545 \quad = \sum_{(\sigma', p')} \left(\sum_{\{(\sigma, p): y = ((\sigma, \sigma'), (p, p')) \in A_1\}} |\Pr_{S^*}[y|\sigma', p'] \Pr_{S^*}[(\sigma', p')] - \Pr_{P^*}[y|\sigma', p'] \Pr_{P^*}[(\sigma', p')] | \right) (*)$$

547 Here

$$548 \quad \Pr_{S^*}[(\sigma', p')] = \sum_{(\sigma, p)} \Pr_{S^*}[(\sigma, \sigma'), (p, p')]$$

549 and

$$550 \quad \Pr_{P^*}[(\sigma', p')] = \sum_{(\sigma, p)} \Pr_{P^*}[(\sigma, \sigma'), (p, p')].$$

551 We define $\delta(\sigma', p') := |\Pr_{S^*}[(\sigma', p')] - \Pr_{P^*}[(\sigma', p')]|$. Let us examine a single term of the
 552 sum $(*)$, for $y = ((\sigma, \sigma'), (p, p'))$:

$$\begin{aligned}
& \left| \Pr_{S^*}[y|\sigma', p'] \Pr_{S^*}[(\sigma', p')] - \Pr_{P^*}[y|\sigma', p'] \Pr_{P^*}[(\sigma', p')] \right| \\
&= \left| \left(\Pr_{S^*}[y|\sigma', p'] \Pr_{S^*}[(\sigma', p')] - \Pr_{P^*}[y|\sigma', p'] \Pr_{S^*}[(\sigma', p')] \right) + \right. \\
&\quad \left. \left(\Pr_{P^*}[y|\sigma', p'] \Pr_{S^*}[(\sigma', p')] - \Pr_{P^*}[y|\sigma', p'] \Pr_{P^*}[(\sigma', p')] \right) \right| \\
&= \left| \left(\Pr_{S_1^*}[(\sigma, p)] - \Pr_{P_1^*}[(\sigma, p)] \right) \Pr_{S^*}[(\sigma', p')] + \Pr_{P_1^*}[(\sigma, p)] \left(\Pr_{S^*}[(\sigma', p')] - \Pr_{P^*}[(\sigma', p')] \right) \right| \\
&\leq \left| \Pr_{S_1^*}[(\sigma, p)] - \Pr_{P_1^*}[(\sigma, p)] \right| \Pr_{S^*}[(\sigma', p')] + \Pr_{P_1^*}[(\sigma, p)] \left| \Pr_{S^*}[(\sigma', p')] - \Pr_{P^*}[(\sigma', p')] \right| \\
&= \left| \Pr_{S_1^*}[(\sigma, p)] - \Pr_{P_1^*}[(\sigma, p)] \right| \Pr_{S^*}[(\sigma', p')] + \Pr_{P_1^*}[(\sigma, p)] \delta(\sigma', p')
\end{aligned}$$

Thus (*) is no more than

$$\begin{aligned}
& \sum_{(\sigma', p')} \sum_{(\sigma, p)} \left| \Pr_{S_1^*}[(\sigma, p)] - \Pr_{P_1^*}[(\sigma, p)] \right| \Pr_{S^*}[(\sigma', p')] \\
&\quad + \sum_{(\sigma', p')} \sum_{\{(\sigma, p): y=((\sigma, \sigma'), (p, p')) \in A_1\}} \Pr_{P_1^*}[(\sigma, p)] \delta(\sigma', p') \\
&\leq \sum_{(\sigma, p)} \left| \Pr_{S_1^*}[(\sigma, p)] - \Pr_{P_1^*}[(\sigma, p)] \right| + \sum_{\{(\sigma', p'): \exists (\sigma, p) ((\sigma, \sigma'), (p, p')) \in A_1\}} \delta(\sigma', p') \\
&= 2\Delta(S_1^*(x), P_1^*(x)) + \sum_{\{(\sigma', p'): \exists (\sigma, p) ((\sigma, \sigma'), (p, p')) \in A_1\}} \delta(\sigma', p') \\
&\leq \frac{2}{2^{|x|}} + \sum_{\{(\sigma', p'): \exists (\sigma, p) ((\sigma, \sigma'), (p, p')) \in A_1\}} \delta(\sigma', p') \quad (**)
\end{aligned}$$

Let us consider a single term $\delta(\sigma', p')$ in the summation in (**). Recalling that the probability that $S(x) = ((\sigma, \sigma'), (p, p'))$ is equal to the probability that $S_1(x) = (\sigma, p)$ and $S_2(x, \sigma, p) = (\sigma', p')$, we have

$$\begin{aligned}
\Pr_{S^*}[(\sigma', p')] &= \sum_{(\sigma, p)} \Pr_{S^*}[((\sigma, \sigma'), (p, p'))] \\
&= \sum_{(\sigma, p)} \Pr_{S^*}[((\sigma, \sigma'), (p, p')) | (\sigma, p)] \Pr_{S^*}[(\sigma, p)] \\
&= \sum_{(\sigma, p)} \Pr_{S_2^*(\sigma, p)}[(\sigma', p')] \Pr_{S_1^*}[(\sigma, p)]
\end{aligned}$$

and similarly $\Pr_{P^*}[(\sigma', p')] = \sum_{(\sigma, p)} \Pr_{P_2^*(\sigma, p)}[(\sigma', p')] \Pr_{P_1^*}[(\sigma, p)]$. Thus

$$\begin{aligned}
575 \quad \delta(\sigma', p') &= \left| \Pr_{S^*}[\sigma', p'] - \Pr_{P^*}[\sigma', p'] \right| \\
576 \quad &= \left| \sum_{(\sigma, p)} \Pr_{S_2^*(\sigma, p)}[(\sigma', p')] \Pr_{S_1^*}[(\sigma, p)] - \sum_{(\sigma, p)} \Pr_{P_2^*(\sigma, p)}[(\sigma', p')] \Pr_{P_1^*}[(\sigma, p)] \right| \\
577 \quad &= \left| \sum_{(\sigma, p)} \Pr_{S_2^*(\sigma, p)}[(\sigma', p')] \Pr_{S_1^*}[(\sigma, p)] - \sum_{(\sigma, p)} \Pr_{P_2^*(\sigma, p)}[(\sigma', p')] \Pr_{P_1^*}[(\sigma, p)] \right| \\
578 \quad &\quad + \sum_{(\sigma, p)} \Pr_{P_2^*(\sigma, p)}[(\sigma', p')] \Pr_{S_1^*}[(\sigma, p)] - \sum_{(\sigma, p)} \Pr_{P_2^*(\sigma, p)}[(\sigma', p')] \Pr_{P_1^*}[(\sigma, p)] \\
579 \quad &= \left| \sum_{(\sigma, p)} \left(\Pr_{S_2^*(\sigma, p)}[(\sigma', p')] - \Pr_{P_2^*(\sigma, p)}[(\sigma', p')] \right) \Pr_{S_1^*}[(\sigma, p)] \right. \\
580 \quad &\quad \left. + \sum_{(\sigma, p)} \Pr_{P_2^*(\sigma, p)}[(\sigma', p')] \left(\Pr_{S_1^*}[(\sigma, p)] - \Pr_{P_1^*}[(\sigma, p)] \right) \right| \\
581 \quad &\leq \sum_{(\sigma, p)} \left| \Pr_{S_2^*(\sigma, p)}[(\sigma', p')] - \Pr_{P_2^*(\sigma, p)}[(\sigma', p')] \right| \Pr_{S_1^*}[(\sigma, p)] \\
582 \quad &\quad + \sum_{(\sigma, p)} \Pr_{P_2^*(\sigma, p)}[(\sigma', p')] \left| \Pr_{S_1^*}[(\sigma, p)] - \Pr_{P_1^*}[(\sigma, p)] \right| \\
583 \quad &= \sum_{(\sigma, p)} 2\Delta(S_2^*(\sigma, p), P_2^*(\sigma, p)) \Pr_{S_1^*}[(\sigma, p)] \\
584 \quad &\quad + \sum_{(\sigma, p)} \Pr_{P_2^*(\sigma, p)}[(\sigma', p')] \left| \Pr_{S_1^*}[(\sigma, p)] - \Pr_{P_1^*}[(\sigma, p)] \right| \\
585 \quad &\leq \sum_{(\sigma, p)} \frac{2}{2^{|x|+p_1(|x|)+q_1(|x|)}} \Pr_{S_1^*}[(\sigma, p)] + \sum_{(\sigma, p)} \Pr_{P_2^*(\sigma, p)}[(\sigma', p')] \left| \Pr_{S_1^*}[(\sigma, p)] - \Pr_{P_1^*}[(\sigma, p)] \right| \\
586 \quad &= \frac{2}{2^{|x|+p_1(|x|)+q_1(|x|)}} + \sum_{(\sigma, p)} \Pr_{P_2^*(\sigma, p)}[(\sigma', p')] \left| \Pr_{S_1^*}[(\sigma, p)] - \Pr_{P_1^*}[(\sigma, p)] \right| \\
587 \quad &
\end{aligned}$$

588 where the last inequality holds, since the summation in (**) is taken over tuples, such
589 that each (x, σ, p) is a YES instance of $L(V_1)$.

590 Replacing each term in (**) with this upper bound, thus yields the following upper bound
591 on (*):

$$\begin{aligned}
592 \quad &\frac{2}{2^{|x|}} + \sum_{(\sigma', p')} \left(\frac{2}{2^{|x|+p_1(|x|)+q_1(|x|)}} + \sum_{(\sigma, p)} \Pr_{P_2^*(\sigma, p)}[(\sigma', p')] \left| \Pr_{S_1^*}[(\sigma, p)] - \Pr_{P_1^*}[(\sigma, p)] \right| \right) \\
593 \quad &= \frac{2}{2^{|x|}} + \frac{2 \cdot 2^{p_2(|x|)+q_2(|x|)}}{2^{|x|+p_1(|x|)+q_1(|x|)}} + \sum_{(\sigma', p')} \sum_{(\sigma, p)} \Pr_{P_2^*(\sigma, p)}[(\sigma', p')] \left| \Pr_{S_1^*}[(\sigma, p)] - \Pr_{P_1^*}[(\sigma, p)] \right| \\
594 \quad &= \frac{2}{2^{|x|}} + \frac{2 \cdot 2^{p_2(|x|)+q_2(|x|)}}{2^{|x|+p_1(|x|)+q_1(|x|)}} + 2\Delta(S_1^*, P_1^*) \\
595 \quad &\leq \frac{2}{2^{|x|}} + \frac{2 \cdot 2^{p_2(|x|)+q_2(|x|)}}{2^{|x|+p_1(|x|)+q_1(|x|)}} + \frac{2}{2^{|x|}} \\
596 \quad &\leq \frac{2}{2^{|x|}} + \frac{2}{2^{|x|}} + \frac{2}{2^{|x|}} \\
597 \quad &
\end{aligned}$$

601 where the last inequality follows from Lemma 28. Thus, A_1 contributes only a negligible
602 quantity to $\Delta(S^*, P^*)$.

603 We now move on to consider A_2 and A_3 .

$$604 \quad \Pr_{P^*}[y \in A_2] = \sum_{\{(\sigma,p):(x,\sigma,p) \in L(V_1)\}} \Pr[V_2(x, \sigma, p) \text{ rejects}] \leq \sum_{(\sigma,p)} \frac{1}{2^{|x|+|\sigma|+|p|}} \leq \frac{1}{2^{|x|}}.$$

$$605 \quad \Pr_{S^*}[y \in A_2] = \sum_{\{(\sigma,p):(x,\sigma,p) \in L(V_1)\}} (\Pr[V_2(x, \sigma, p) \text{ rejects}] + \Delta(S_2^*(\sigma, p), P_2^*(\sigma, p))) \leq \frac{2}{2^{|x|}}.$$

606 A similar and simpler calculation shows that $\Pr_{P^*}[y \in A_3] \leq \frac{1}{2^{|x|}}$ and $\Pr_{S^*}[y \in A_3] \leq \frac{2}{2^{|x|}}$,
607 to complete the proof.

608 ◀

609 ► **Corollary 29.** $\text{NISZK}_L = \text{NISZK}_{AC^0} = \text{NISZK}_{AC^0, \text{DET}} = \text{NISZK}_{NL, \text{DET}}$

610 The proof of Theorem 27 did not make use of the condition that the verifier is at least as
611 powerful as the simulator. Thus, maintaining the condition that $A \subseteq B \subseteq \text{NISZK}_A$, we also
612 have the following corollary:

613 ► **Corollary 30.** $\text{NISZK}_B = \text{NISZK}_{B,A}$

614 ► **Corollary 31.** $\text{NISZK}_{A,B} \subseteq \text{NISZK}_{B,A}$

615 ► **Corollary 32.** $\text{NISZK}_{\text{DET}} = \text{NISZK}_{\text{DET}, AC^0}$

616 7 SZK_L closure under $\leq_{\text{bf-tt}}^L$ reductions

617 Although our focus in this paper has been on NISZK_L , in this section we report on a closure
618 property of the closely-related class SZK_L .

619 The authors of [14], after defining the class SZK_L , wrote:

620 We also mention that all the known closure and equivalence properties of SZK (e.g.
621 closure under complement [24], equivalence between honest and dishonest verifiers
622 [17], and equivalence between public and private coins [24]) also hold for the class
623 SZK_L .

624 In this section, we consider a variant of a closure property of SZK (closure under $\leq_{\text{bf-tt}}^P$
625 [27]), and show that it also holds⁷ for SZK_L . Although our proof follows the general approach
626 of the proof of [27, Theorem 4.9], there are some technicalities with showing that certain
627 computations can be accomplished in logspace (and for dealing with distributions represented
628 by branching programs instead of circuits) that require proof. (The characterization of SZK_L
629 in terms of reducibility to the Kolmogorov-random strings presented in [4, Theorem 34] relies
630 on this closure property.)

⁷ We observe that open questions about closure properties of NISZK also translate to open questions about NISZK_L . NISZK is not known to be closed under union [25], and neither is NISZK_L . Neither is known to be closed under complementation. Both are closed under conjunctive logspace-truth-table reductions.

631 ► **Definition 33.** (From [27, Definition 4.7]) For a promise problem Π , the characteristic
 632 function of Π is the map $\mathcal{X}_\Pi : \{0, 1\}^* \rightarrow \{0, 1, *\}$ given by

$$633 \quad \mathcal{X}_\Pi(x) = \begin{cases} 1 & \text{if } x \in \Pi_{Yes}, \\ 0 & \text{if } x \in \Pi_{No}, \\ * & \text{otherwise.} \end{cases}$$

634 ► **Definition 34.** Logspace Boolean formula truth-table reduction ($\leq_{\text{bf-tt}}^L$ reduction): We
 635 say a promise problem Π **logspace Boolean formula truth-table reduces** to Γ if there
 636 exists a logspace-computable function f , which on input x produces a tuple (y_1, \dots, y_m) and
 637 a Boolean formula ϕ (with m input gates) such that:

$$638 \quad x \in \Pi_{Yes} \implies \phi(\mathcal{X}_\Gamma(y_1), \dots, \mathcal{X}_\Gamma(y_m)) = 1$$

$$639 \quad x \in \Pi_{No} \implies \phi(\mathcal{X}_\Gamma(y_1), \dots, \mathcal{X}_\Gamma(y_m)) = 0$$

641 We begin by proving a logspace analogue of a result from [27], used to make statistically
 642 close pairs of distributions closer and statistically far pairs of distributions farther.

643 ► **Lemma 35.** (Polarization Lemma, adapted from [27, Lemma 3.3]) There is a logspace-
 644 computable function that takes a triple $(P_1, P_2, 1^k)$, where P_1 and P_2 are branching programs,
 645 and outputs a pair of branching programs (Q_1, Q_2) such that:

$$646 \quad \Delta(P_1, P_2) < \frac{1}{3} \implies \Delta(Q_1, Q_2) < 2^{-k}$$

$$647 \quad \Delta(P_1, P_2) > \frac{2}{3} \implies \Delta(Q_1, Q_2) > 1 - 2^{-k}$$

649 To prove this, we adapt the same method as in [27] and alternate two different procedures,
 650 one to drive pairs with large statistical distance closer to 1, and one to drive distributions
 651 with small statistical distance closer to 0. The following lemma will do the former:

652 ► **Lemma 36.** (Direct Product Lemma, from [27, Lemma 3.4]) Let X and Y be distributions
 653 such that $\Delta(X, Y) = \epsilon$. Then for all k ,

$$654 \quad k\epsilon \geq \Delta(\otimes^k X, \otimes^k Y) \geq 1 - 2\exp(-k\epsilon^2/2)$$

655 The proof of this statement follows from [27]. To use this for Lemma 35, we note that a
 656 branching program for $\otimes^k P$ can easily be created in logspace from a branching program P
 657 by simply copying and concatenating k independent copies of P together.

658 We now introduce a lemma to push close distributions closer:

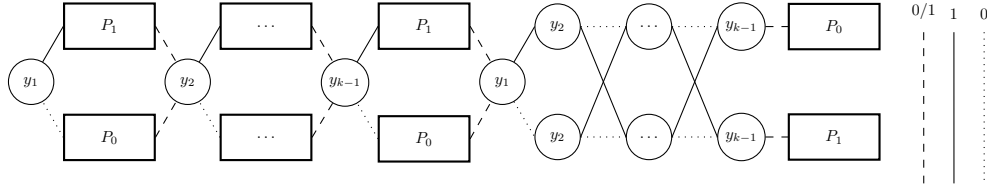
659 ► **Lemma 37.** (XOR Lemma, adapted from [27, Lemma 3.5]) There is a logspace-computable
 660 function that maps a triple $(P_0, P_1, 1^k)$, where P_0 and P_1 are branching programs, to a pair
 661 of branching programs (Q_0, Q_1) such that $\Delta(Q_0, Q_1) = \Delta(P_0, P_1)^k$. Specifically, Q_0 and Q_1
 662 are defined as follows:

$$663 \quad Q_0 = \bigotimes_{i \in [k]} P_{y_i} : y \sim \{y \in \{0, 1\}^k : \oplus_{i \in [k]} y_i = 0\}$$

$$664 \quad Q_1 = \bigotimes_{i \in [k]} P_{y_i} : y \sim \{y \in \{0, 1\}^k : \oplus_{i \in [k]} y_i = 1\}$$

Proof. The proof that $\Delta(Q_0, Q_1) = \Delta(P_0, P_1)^k$ follows from [27, Proposition 3.6]. To finish proving this lemma, we show a logspace-computable mapping between $(P_0, P_1, 1^k)$ and (Q_0, Q_1) .

Let ℓ and w be the max length and width between P_0 and P_1 . We describe the structure of Q_0 , with Q_1 differing in a small step: to begin with, Q_0 reads the $k - 1$ random bits y_1, \dots, y_{k-1} . For each of the random bits, it can pick the correct of two different branches, one having P_0 built in at the end and the other having P_1 . We will read y_1 , branch to P_0 or P_1 (and output the distribution accordingly), then unconditionally branch to reading y_2 and repeat until we reach y_{k-1} and branch to P_0 or P_1 . We then unconditionally branch to y_1 and start computing the parity, and at the end we will be able to decide the value of y_k which will allow us to branch to the final copy of P_0 or P_1 .



■ **Figure 1** Branching program for Q_0 of Lemma 37

Creating (Q_0, Q_1) can be done in logspace, requiring logspace to create the section to compute y_k and logspace to copy the independent copies of P_0 and P_1 .

We now have the tools to prove Lemma 35.

Proof. (of Lemma 35) From [27, Section 3.2], we know that we can polarize $(P_0, P_1, 1^k)$ by:

- Letting $l = \lceil \log_{4/3} 6k \rceil$, $j = 3^{l-1}$
- Applying Lemma 37 to $(P_0, P_1, 1^l)$ to get (P'_0, P'_1)
- Applying Lemma 36: $P''_0 = \otimes^j P'_0$, $P''_1 = \otimes^j P'_1$
- Applying Lemma 37 to $(P''_0, P''_1, 1^k)$ to get (Q_0, Q_1)

Each step is computable in logspace, and since logspace is closed under composition, this completes our proof.

We also mention the following lemma, which will be useful in evaluating the Boolean formula given by the $\leq_{\text{bf-tt}}^L$ reduction.

► **Lemma 38.** *There is a function in NC^1 that takes as input a Boolean formula ϕ (with m input bits) and produces as output an equivalent formula ψ with the following properties:*

1. *The depth of ψ is $O(\log m)$.*
2. *ψ is a tree with alternating levels of AND and OR gates.*
3. *The tree's non-leaf structure is always the same for a fixed input length, and is a complete binary tree.*
4. *All NOT gates are located just before the leaves.*

Proof. Although this lemma does not seem to have appeared explicitly in the literature, it is known to researchers, and is closely related to results in [15] (see Theorems 5.6 and 6.3, and Lemma 3.3) and in [5] (see Lemma 5). Alternatively, one can derive this by using the fact that the Boolean formula evaluation problem lies in NC^1 [10, 11], and thus there is

an alternating Turing machine M running in $O(\log n)$ time that takes as input a Boolean formula ψ and an assignment α to the variables of ψ , and returns $\psi(\alpha)$. We may assume without loss of generality that M alternates between existential and universal states at each step, and that M runs for exactly $c \log n$ steps on each path (for some constant c), and that M accesses its input (via the address tape that is part of the alternating Turing machine model) only at a halting step, and that M records the sequence of states that it has visited along the current path in the current configuration. Thus the configuration graph of M , on inputs of length n , corresponds to a formula of $O(\log n)$ depth having the desired structure, and this formula can be constructed in NC^1 . Given a formula ϕ , an NC^1 machine can thus build this formula, and hardwire in the bits that correspond to the description of ϕ , and identify the remaining input variables (corresponding to M reading the bits of α) with the variables of ϕ . The resulting formula is equivalent to ϕ and satisfies the conditions of the lemma. \blacktriangleleft

► **Definition 39.** (From [27, Definition 4.8]) For a promise problem Π , we define a new promise problem $\Phi(\Pi)$ as follows:

$$\Phi(\Pi)_{Yes} = \{(\phi, x_1, \dots, x_m) : \phi(\mathcal{X}_\Pi(x_1), \dots, \mathcal{X}_\Pi(x_m)) = 1\}$$

$$\Phi(\Pi)_{No} = \{(\phi, x_1, \dots, x_m) : \phi(\mathcal{X}_\Pi(x_1), \dots, \mathcal{X}_\Pi(x_m)) = 0\}$$

► **Theorem 40.** SZK_L is closed under $\leq_{\text{bf-tt}}^L$ reductions.

To begin the proof of this theorem, we first note that as in the proof of [27, Lemma 4.10], given two SD_{BP} pairs, we can create a new pair which is in $\text{SD}_{\text{BP}, No}$ if both of the original two pairs are (which we will use to compute ANDs of queries.) We can also compute in logspace the OR query for two queries by creating a pair $(P_1 \otimes S_1, P_2 \otimes S_2)$. We prove that these operations produce an output with the correct statistical difference with the following two claims:

▷ **Claim 41.** $\{(y_1, y_2) | \mathcal{X}_{\text{SD}_{\text{BP}}}(y_1) \vee \mathcal{X}_{\text{SD}_{\text{BP}}}(y_2) = 1\} \leq_m^L \text{SD}_{\text{BP}}$.

Proof. Let $y_1 = (A_1, B_1)$ and $y_2 = (A_2, B_2)$. Let $p > 0$ be a parameter, where we are guaranteed that:

$$(A_i, B_i) \in \text{SD}_{\text{BP}, Y} \implies \Delta(A_i, B_i) > 1 - p$$

$$(A_i, B_i) \in \text{SD}_{\text{BP}, N} \implies \Delta(A_i, B_i) < p$$

Then consider:

$$y = (A_1 \otimes A_2, B_1 \otimes B_2)$$

Let us analyze the Yes and No instance of $\mathcal{X}_{\text{SD}_{\text{BP}}}(y_1) \vee \mathcal{X}_{\text{SD}_{\text{BP}}}(y_2)$:

- YES: $\Delta(A_1 \otimes A_2, B_1 \otimes B_2) \geq \max\{\Delta(A_1 \otimes B_2, B_1 \otimes B_2), \Delta(B_1 \otimes A_2, B_1 \otimes B_2)\} = \max\{\Delta(A_1, B_1), \Delta(A_2, B_2)\} > 1 - p$.
- NO⁸: $\Delta(A_1 \otimes A_2, B_1 \otimes B_2) \leq \Delta(A_1, B_1) + \Delta(A_2, B_2) < 2p$.

⁸ For the first inequality here, see [27, Fact 2.3].

In our Boolean formula, we will have only $d = O(\log m)$ depth, so we have this OR operation for at most $\frac{d+1}{2}$ levels (and the soundness gap doubles at every level). Since $p = \frac{1}{2^m}$ at the beginning, the gap (for NO instance) will be upper bounded at the end by:

$$< 2^{\frac{d+1}{2}} \frac{1}{2^m} = \frac{m^{O(1)}}{2^m} < 1/3.$$

▷ **Claim 42.** $\{(y_1, y_2) \mid \mathcal{X}_{\text{SD}_{\text{BP}}}(y_1) \wedge \mathcal{X}_{\text{SD}_{\text{BP}}}(y_2) = 1\} \leq_m^{\text{L}} \text{SD}_{\text{BP}}$.

Proof. Let $y_1 = (A_1, B_1)$ and $y_2 = (A_2, B_2)$. Let $p > 0$ be a parameter, where we are guaranteed that:

$$(A_i, B_i) \in \text{SD}_{\text{BP}, Y} \implies \Delta(A_i, B_i) > 1 - p$$

$$(A_i, B_i) \in \text{SD}_{\text{BP}, N} \implies \Delta(A_i, B_i) < p$$

We can construct a pair of BPs $y = (A, B)$ whose statistical difference is exactly

$$\Delta(A_1, B_1) \cdot \Delta(A_2, B_2)$$

The pair (A, B) we construct is analogous to (Q_0, Q_1) in Lemma 37, and can be created in logspace with 2 random bits b_0, b_1 . We have $A = (A_1, A_2)$ if $b_0 = 0$ and $A = (B_1, B_2)$ if $b_0 = 1$, while $B = (A_1, B_2)$ if b_2 is 0 and (A_2, B_1) if $b_1 = 1$.

Let us analyze the Yes and No instance of $\mathcal{X}_{\text{SD}_{\text{BP}}}(y_1) \wedge \mathcal{X}_{\text{SD}_{\text{BP}}}(y_2)$:

- YES: $\Delta(A_1, B_1) \cdot \Delta(A_2, B_2) > (1 - p)^2$.
- NO: $\Delta(A_1, B_1) \cdot \Delta(A_2, B_2) \leq \max\{\Delta(A_1, B_1), \Delta(A_2, B_2)\} < p$.

◀

In our Boolean formula we will have only $d = O(\log m)$ depth, so we have this AND operation for at most $\frac{d+1}{2}$ levels (and the completeness gap squares itself at every level). Since $p = \frac{1}{2^m}$ at the beginning, the gap (for YES instance) will be lower bounded at the end by:

$$> (1 - \frac{1}{2^m})^{2^{\frac{d+1}{2}}} = (1 - \frac{1}{2^m})^{m^{O(1)}} > (1 - \frac{1}{2^m})^{2^m/m} \approx (\frac{1}{e})^{1/m} > \frac{2}{3}.$$

Proof. (of Theorem 40) Now suppose that we are given a promise problem Π such that $\Pi \leq_{\text{bf-tt}}^{\text{L}} \text{SD}_{\text{BP}}$. We want to show $\Pi \leq_m^{\text{L}} \text{SD}_{\text{BP}}$, which by SZK_{L} 's closure under \leq_m^{L} reductions implies $\Pi \in \text{SZK}_{\text{L}}$.

We follow the steps below on input x to create an SD_{BP} instance (F_0, F_1) which is in $\text{SD}_{\text{BP}, Y}$ if $x \in \Pi_Y$, and is in $\text{SD}_{\text{BP}, N}$ if $x \in \Pi_N$:

1. Run the L machine for the $\leq_{\text{bf-tt}}^{\text{L}}$ reduction on x to get queries (q_1, \dots, q_m) and the formula ϕ .
2. Build ψ from ϕ using Lemma 38. Recalling that there is a \leq_m^{L} reduction f reducing SD_{BP} to its complement, replace each negated query $\neg q_i$ with $f(q_i)$, so that we can now view ψ as a *monotone* Boolean formula reducing Π to SD_{BP} . Since the Polarization Lemma (Lemma 35) maps YES instances to YES instances and NO instances to NO instances, we can also use the same formula ψ on the polarized instances that we obtain by applying Lemma 35 with $k = n$ to these queries, to obtain a new list of queries (y_1, \dots, y_m) . Furthermore we may pad these queries, so that each query y_i consists of a pair of branching programs (instances of SD_{BP}) where all of the branching programs have the same number of output bits.

- 778 3. Using the formula ψ , build a “template tree” T . At the leaf level, for each variable in ψ ,
 779 we will plug in the corresponding query y_i ; interior nodes are labeled AND or OR. By
 780 Lemma 38 the tree T is full. Using Claims 41 and 42, each node of the template tree is
 781 associated with a pair of branching programs, with the pair (F_0, F_1) at the root being the
 782 output of our \leq_m^L reduction. It is important to note that the constructions in Claims 41
 783 and 42 produce distributions, where each output bit is simply a copy of one of the output
 784 bits of the distributions that feed into it. Thus each output bit of F_0 and F_1 is simply a
 785 copy of one of the output bits of one of the pairs of branching programs that constitute
 786 one of the input queries y_i .
- 787 4. Given x and designated output position j of F_0 or F_1 , there is a logspace computation
 788 which finds the original output bit from $y_1 \dots y_m$ that bit j was copied from. This machine
 789 traverses down the template tree from the output bit and records the following:
- 790 – The node that the computation is currently at on the template tree, with the path
 791 taken depending on j .
 - 792 – The position of the random bits used to decide which path to take when we reach
 793 nodes corresponding to AND.

794 This takes $O(\log m)$ space. We can use this algorithm to copy and compute each output
 795 bit of F_0 and F_1 , creating (F_0, F_1) in logspace.

796 For step 4, we give an algorithm $\text{Eval}(x, j, \psi, y_1, \dots, y_m)$ to compute the j th output bit of
 797 F_0 or F_1 on x , for a formula ψ satisfying the properties of Lemma 38, a list of SD_{BP} queries
 798 (y_1, \dots, y_m) , and j . Without loss of generality, we lay out the algorithm to compute only
 799 $F_0(x)$.

800 Outline of $\text{Eval}(x, j, \psi, y_1, \dots, y_m)$:

801 The idea is to compute the j th output bit of F_0 by recursively calculating which query
 802 output bit it was copied from. To do this, first notice that the AND and OR operations
 803 produce branching programs where each output bit is copied from exactly one output bit of
 804 one of the query branching programs, so composing these operations together tells us that
 805 every output bit in F_0 is copied from exactly one output bit from one query. By Lemma 38
 806 and our AND and OR operations preserving the number of output bits, we also have that
 807 if every BP has l output bits, F_0 will have $2^a l = |\psi|l$ output bits, where a is the depth of
 808 ψ . This can be used to recursively calculate which query the j th bit is from: for an OR
 809 gate, divide the output bits into fourths, and decide which fourth the j th bit falls into (with
 810 each fourth corresponding to one BP, or two fourths corresponding to a subtree.) For an
 811 AND gate, divide the output into fourths, decide which fourth the j th bit falls into, and
 812 then use the 4 random bits for the XOR operation to compute which fourth corresponds to
 813 which branching programs (2 fourths will correspond to 1 BP or subtree, and the other 2
 814 fourths will correspond to the 2 BPs from the other subtree.) If j is updated recursively,
 815 then at the query level, we can directly return the j' th output bit. This can be done in
 816 logspace, requiring a logspace path of “lefts” and “rights” to track the current gate, logspace
 817 to record and update j' , logspace to compute $2^a l$ at each level, and logspace to compute
 818 which subtree/query the output bit comes from at each level.

819 The resulting BP will be two distributions that will be in $\text{SD}_{\text{BP}, Y} \iff x \in \Pi_Y$. By this
 820 process $\Pi \leq_m^L \text{SD}_{\text{BP}}$. ◀

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